Error boundedness of Correction Procedure via Reconstruction / Flux Reconstruction

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We study the long time error behaviour of the correction procedure via reconstruction / flux reconstruction methods to hyperbolic conservation laws. We show that not only the choice of the numerical flux (upwind or central) affects the growth rate and asymptotic value of the error, but also that the selection of bases (Gauß-Lobatto or Gauß-Legendre) is even more important. Using Gauß-Legendre basis, the error reaches the asymptotic value faster, and to a lower value than by using Gauß-Lobatto basis. Also the differences in the error caused by the numerical flux is not essential for low resolution computations in the Gauß-Legendre case.

1 Introduction

There exist plenty of examples in the literature, where stable approximations of hyperbolic conservation laws demonstrate a linear growth (or near linear growth) in time, even though stability of the numerical schemes should guarantee that the solution remains bounded, see [5, 9, 16]. The reason behind this is the following: The error equation for the time variation contains a forcing term generated by the approximation or truncation errors and this forcing term can trigger the unbounded growth of the error.

Simultaneously, there are other examples, where the temporal error growth is bounded [1, 10] and finally, in [14] the author gives an explanation under what conditions the error is or is not bounded in time. He works with the SBP-SAT (Summation-by-Parts/Simultaneous-Approximation-Term) finite difference approximations and summarises, that it depends only on the choice of boundary procedure of the problem. If one catches the waves in cavities or in periodic boundary conditions, than linear growth is observed like it is investigated in [9], whereas for inflow-outflow problems one obtains boundedness. In other words, if an appropriate boundary approach (sufficiently dissipative) is applied, the error is bounded. In this framework it does not depend on the internal discretisation. In [12], the authors analyse the long time behaviour of the error for the discontinuous Galerkin spectral element methods (DGSEM). They confirm the conclusion from [14] that the bounded error property is due to the fact of dissipative boundary conditions, but different from [14], in the DGSEM framework the internal approximation has indeed an influence on the behaviour of the error. The choice of the numerical flux (upwind or central) is essential for the magnitude of the error.
and the speed at which the asymptotic error is reached. With the upwind flux one obtains better results. In this paper we examine the long time error behaviour for the recent correction procedure via reconstruction (CPR) / flux reconstruction (FR) methods. The CPR/FR is a unifying framework for several high-order methods such as discontinuous Galerkin (DG), spectral difference (SD) and spectral volume (SV) methods and also includes by a special choice of the nodal basis and the correction matrix the DGSEM of [12]. Here, we investigate not only the numerical flux, but also the selection of nodal bases (Gauß-Legendre and Gauß-Lobatto). We recognise that the selection of the flux function is less important as the choice of the nodal bases for the error behaviour. Using Gauß-Legendre basis in the approximation, the error is lower than in the Gauß-Lobatto case and the speed of attaining the error asymptotic is even faster for most of the problems under consideration. The selection of the numerical flux takes less influence on the error behaviour in the Gauß-Legendre case as applying a Gauß-Lobatto basis.

The paper is organised as follows: In the first section, we repeat the main ideas of the SBP-CPR/FR methods. Then, we present the model problem under consideration. In the next section 4, we concentrate on the SBP-CPR/FR formulation, which is equivalent to the DGSEM, but we extend our investigation to Gauß-Legendre basis. Gauß-Legendre nodes do not contain the boundary values in one element and this yields to a further error term in our error equation. We focus on this additional error term and give an interpretation for it. Afterwards, we transfer and extend the results about the long time error behaviour from the special case in section 4 to the linear stable one-parameter family of Vincent [23, 24]. Here, we consider more general correction terms and norms. We demonstrate our investigation by numerical tests 6, which include also one example from [12] for comparison. We mention the limitations of our results and finally, we summarise and discuss these limitations critically.

2 Correction Procedure via Reconstruction/Flux Reconstruction using Summation-by-Parts operators

We will shortly repeat the main idea of the CPR/FR methods for a scalar, one-dimensional hyperbolic conservation law
\[ \partial_t u + \partial_x f(u) = 0, \] (1)
equipped with adequate initial and boundary conditions. The domain \( \Omega \subset \mathbb{R} \) is split into disjoint open intervals \( \Omega_i \subset \Omega \) such that \( \bigcup_i \Omega_i = \Omega \). The CPR/FR method is a semidiscretisation applying a polynomial approximation on elements. Each element \( \Omega_i \) is transferred onto a standard element, which is in our case simply \((-1, 1)\). All calculations are conducted within this standard element. Let \( \mathbb{P}_N \) be the space of polynomials of degree \( \leq N \) and \( \mathbb{I}^N : L^2(-1,1) \to \mathbb{P}_N(-1,1) \) be the interpolation operator. The solution \( u \) is approximated by a polynomial \( U \in \mathbb{P}_N \) and in the basic formulation a nodal Lagrange basis is employed. The coefficients \( u \) of \( u \) are given by the nodal values \( u_i = u(\zeta_i), i \in \{0,\ldots,N\} \), where \(-1 \leq \zeta_i \leq 1\) are interpolation points in \([-1,1]\). It can be written as
\[ u \approx U = \sum_{i=0}^{N} u_i l_i(\xi), \] (2)
where \( l_i(\xi) \) is the \( i \)-th Lagrange interpolation polynomial that satisfies \( l_j(\xi_j) = \delta_{ij} \). The flux \( f(u) \) is also approximated by a polynomial, where the coefficients are given by \( f_i = f(u_i) = f(u(\zeta_i)) \). We apply a discrete derivative matrix \( D \) on \( f \). The divergence is \( Df \). Since the solutions will probably have discontinuities across elements, we will have this in the discrete flux, too. In order to avoid this problem, we introduce a numerical flux \( f_{num} \) and also a correction term \( C \) at the boundary nodes [21]. Hence, the CPR/FR method reads
\[ \partial_t u = -Df - C \left( f_{num} - R f \right), \] (3)
where the restriction matrix \( R \) performs interpolation to the boundary. A general choice of the correction matrix \( C \) recovers the linearly stable flux reconstruction methods of [23, 24], as described by [21]. The
canonical choice is
\[ C = M^{-1} R^T B, \]  
which is a generalisation of SATs used in finite difference methods and corresponds to a strong form of the discontinuous Galerkin method [11].

The numerical flux \( f_{\text{num}} \) computes a common flux on the boundary using values from both neighbouring elements. With respect to a chosen basis, the scalar product approximating the \( L^2 \) scalar product is represented by a matrix \( M \) and integration with respect to the outer normal by \( B \). Finally, all operators are introduced and they have to fulfil the SBP property
\[ M D + D^T M = R^T B R, \]  
in order to mimic integration by parts on a discrete level
\[ u^T M D v + u^T D^T M v \approx \int_{\Omega} (\partial_x u)v + \int_{\partial\Omega} (\partial_n u)v \mid_{\partial\Omega} \approx u^T R^T B R v. \]  

Different bases can be used like nodal Gauß-Legendre / Gauß-Lobatto-Legendre or modal Legendre bases, as described in [22]. As an example, in the standard element \([-1,1]\] the SBP-CPR method for the linear advection equation can be formulated as
\[ u_t + D u + C (f_{\text{num}} - R u) = 0. \]  

3 Model problem

To analyse the long time error behaviour of the SBP-CPR/FR methods, we study analogue to [12, 14] the scalar linear advection equation with non-periodic boundary conditions
\[ \begin{align*}
    u_t + u_x &= 0, \quad x \in [0,L], \quad t \geq 0 \\
    u(t,0) &= g(t), \\
    u(0,x) &= u_0(x).
\end{align*} \]  

We assume also that the initial and boundary values to be chosen in such way that \( u(t,x) \in H^m(0,L) \) for \( m > 1 \) and that its norm \( ||u||_{H^m} \) is uniformly bounded in time. Like it is described in [12], such conditions are physically meaningful, because they describe problems where the boundary input is, for instance, sinusoidal. In our numerical tests in section 6, we will present also an example, where these conditions are not fulfilled, see subsection 6.2.

We will shortly repeat the aspect from [12], why the impact of boundary condition on the solution is essential. The energy of the solution \( u \) of our initial boundary value problem (8) is measured by the \( L^2 \)-norm \( ||u||^2 = \int_0^L u^2 \, dx \). Focusing on the weak formulation of the advection equation (8), we multiply with a test function \( \varphi \in C^1(0,L) \) and integrate over the domain. We get
\[ \int_0^L u_t \varphi \, dx + \int_0^L u_x \varphi \, dx = 0. \]
\[ \varphi = u \text{ and integration by parts yield} \]
\[ \frac{1}{2} \frac{d}{dt} ||u||^2 = - \int_0^L u u_x \, dx = g^2(t) - u^2(L,t). \]

Integration in time over an interval \([0,T]\) leads to
\[ ||u(T)||^2 + \int_0^T u^2(L,t) \, dt = ||u_0||^2 + \int_0^T g^2(t) \, dt. \]
Here, we see that the energy at time $T$ can be expressed by the initial energy plus the energy added at the left side through the boundary condition minus the energy, which we lose through the right side. Therefore, the selection of the boundary conditions is substantial and the numerical approximation has to imitate this.

In [15], the author shows that a correct implementation of the boundary conditions is essentially for well posedness.

4 Long time error behaviour for SBP-CPR/FR

4.1 Stability of the SBP-CPR/FR methods

We follow the steps from [12] and focus first on stability of the SBP-CPR/FR methods. Afterwards, we derive an error equation for the SBP-CPR/FR methods to the model problem (8). Different from [12], we also consider Gauß-Lobatto and Gauß-Legendre quadrature points in this context, whereas we focus on the one-parameter family of Vincent et al. [23] in the next section 5. We analyse stability in the semidiscrete sense. Therefore, we divide first of all the interval in elements $e^k = [x_k, x_{k+1}], k = 1, \ldots, K$, whereas the $x_k$ are the element boundaries with $x_0 = 0$ and $x_K = L$. Like it was explained in section 2, we transform every element to our standard element and use here the SBP-CPR/FR method. We investigate both Gauß-Lobatto and Gauß-Legendre quadrature. We also provide estimations for a modal Legendre basis. Here, we would assume exact integration. Then, the analysis for modal Legendre basis is similar to Gauß-Legendre and can be transferred with these estimations.

We can define the discrete inner product by

- Gauß-Lobatto

$$ (U, V)_N := \sum_{j=0}^N U(\xi_j)V(\xi_j)\omega_j = \int_{-1}^1 UV \, d\xi \quad \forall UV \in \mathbb{P}^{2N-1}, $$

(10)

- Gauß-Legendre

$$ (U, V)_N := \sum_{j=0}^N U(\xi_j)V(\xi_j)\omega_j = \int_{-1}^1 UV \, d\xi \quad \forall UV \in \mathbb{P}^{2N+1}. $$

(11)

We choose the numerical flux to have the form

$$ f_{\text{num}}(U_L, U_R) = \frac{U_L + U_R}{2} - \frac{\sigma}{2}(U_R - U_L), \quad \sigma \in [0, 1], $$

where $U_L, U_R$ are the states on the left and right boundaries. Considering now the model problem (8). To study stability for the SBP-CPR/FR method on one element, we transform the SBP-CPR/FR formulation of the advection equation (7) into a weak formulation. The numerical flux is given by $f_{\text{num},k} = (f_{L,k} \cdot f_{R,k})^T$. Therefore, we multiply $\varphi^{k,T}M$ to (7). Here, $k$ describes the element and $T$ means only the transposed vector. We get

$$ \frac{\Delta x_k}{2} \varphi^{k,T}M u^k + \varphi^{k,T}M D u^k + \varphi^{k,T}M C \left( f_{\text{num},k} - R u^k \right) = 0, $$

(12)

with the length of the element $\Delta x_k = x_k - x_{k-1}$. $\frac{\Delta x_k}{2}$ is a transformation factor, because we calculate everything in our standard element.

Using the SBP-property (5) and $C = M^{-1}R^T B$ yields

$$ \frac{\Delta x_k}{2} \varphi^{k,T}M u^k + \varphi^{k,T}M D u^k + \varphi^{k,T}M M^{-1}R^T B \left( f_{\text{num},k} - R u^k \right) = 0, $$

$$ \frac{\Delta x_k}{2} \varphi^{k,T}M u^k + \varphi^{k,T}D \left( R^T B \right) u^k + \varphi^{k,T}D M u^k + \varphi^{k,T}D M u^k + \varphi^{k,T}D R^T B \left( R u^k \right) = 0, $$

(13)

$$ \frac{\Delta x_k}{2} \varphi^{k,T}M u^k + \varphi^{k,T}D \left( R^T B \right) u^k + \varphi^{k,T}D M u^k + \varphi^{k,T}D R^T B \left( R u^k \right) = 0. $$
This can be reformulate, by using (2) and the Gauß-Lobatto / Gauß-Legendre quadrature fulfilling, as (10)-(11)

\[ \frac{\Delta x_k}{2} \left( \partial_t U^k, \varphi^k \right)_N + \sum_{i=L,R} u^{k,T} R^T B_{R}^{i} u^{\text{num},k} - \left( u^{k}, \partial_t \varphi^k \right)_N = 0. \] (14)

For the stability investigation, we work with the SBP-CPR/FR formulation from (13) and put \( \varphi^k = u^k \). It is

\[ \frac{\Delta x_k}{4} \frac{d}{dt} ||u^k||_N^2 = - \left( u^{k,T} R^T B R u^k - u^{k,T} M u^k \right). \]

Using the SBP property (5) and the symmetry give

\[ u^{k,T} D^T M u^k = u^{k,T} R^T B R u^k - u^{k,T} M u^k, \]

\[ \Leftrightarrow u^{k,T} D^T M u^k = \frac{1}{2} u^{k,T} R^T B R u^k. \]

Together, we obtain

\[ \frac{\Delta x_k}{4} \frac{d}{dt} ||u^k||_N^2 = - u^{k,T} R^T B \left( f^{\text{num},k} - \frac{1}{2} M u^k \right). \] (15)

This is the rate of change of the energy in on element as it was already studied in [21]. The rate of change of the total energy is the sum over all elements,

\[ \frac{1}{2} \frac{d}{dt} \sum_{k=1}^{K} \frac{\Delta x_k}{2} ||u^k||_N^2 = - \sum_{k=1}^{K} u^{k,T} R^T B \left( f^{\text{num},k} - \frac{1}{2} M u^k \right). \] (16)

We have to be careful at the left and the right physical boundaries. Therefore, we split this sum into three parts and apply also the fact, that the numerical flux is unique at the interface of two elements.

\[ \sum_{k=1}^{K} u^{k,T} R^T B \left( f^{\text{num},k} - R u^k \right) = u^{1,T} R^T B \left( f^{\text{num},1} - \frac{1}{2} R u^1 \right) \]

\[ + \sum_{k=2}^{K-1} u^{k,T} R^T B \left( f^{\text{num},k} - R u^k \right) + u^{K,T} R^T B \left( f^{\text{num},K} - \frac{1}{2} R u^K \right) \]

\[ = -U_L^1 \left( f^{\text{num},1} - \frac{1}{2} U_L^1 \right) + U_R^1 \left( f^{\text{num},1} - \frac{1}{2} U_R^1 \right) + \sum_{k=2}^{K-1} U_R^k \left( f^{\text{num},k} - \frac{1}{2} U_R^k \right) \]

\[ -U_L^k \left( f^{\text{num},k} - \frac{1}{2} U_L^k \right) + U_K^k \left( f^{\text{num},K} - \frac{1}{2} U_K^k \right) - U_L^K \left( f^{\text{num},K} - \frac{1}{2} U_L^K \right). \]

For both Gauß-Legendre and Gauß-Lobatto the discrete product \( u^{k,T} D^T M u^k \) is exact. We describe with \( U_i \) \((i=L,R)\) the approximated solution (2) and the indices give the position in the elements. The numerical flux \( f_i^{\text{num},k} \) is also evaluated on the left or right boundaries. We may combine the equation in the following form:

\[ \sum_{k=1}^{K} u^{k,T} R^T B \left( f^{\text{num},k} - R u^k \right) = -U_L^1 \left( f^{\text{num},1} - \frac{1}{2} U_L^1 \right) \]

\[ + \sum_{k=2}^{K} \left( f^{\text{num},k} \left( U_R^{k-1} - U_L^k \right) - \frac{1}{2} \left( U_R^{k-1} + U_L^k \right) \right) \]

\[ + U_R^K \left( f^{\text{num},K} - \frac{1}{2} U_R^K \right). \]
For calculating the numerical flux \( f_{L}^{num} = f_{L}^{num}(g, U_{L}^{1}) \), we apply the boundary condition \( g \), whereas we need for \( f_{R}^{num,K} = f_{R}^{num,K}(U_{R}^{K}, U_{ext}) \) some external state \( U_{ext} \). If we choose now the upwind flux, we get

\[
U_{L}^{1} \left( f_{num}^{1} - \frac{1}{2} U_{L}^{1} \right) = U_{L}^{1} \left( g_{num}(g, U_{L}^{1}) - \frac{1}{2} U_{L}^{1} \right) = \frac{1}{2} g^{2} - \frac{1}{2} \left( U_{L}^{1} - g \right)^{2},
\]

\[
U_{R}^{K} \left( f_{num}^{num,K} - \frac{1}{2} U_{R}^{K} \right) = U_{R}^{K} \left( f_{num,K}(U_{R}^{K}, U_{ext}) - \frac{1}{2} U_{R}^{K} \right) = \frac{1}{2} \left( U_{R}^{K} \right)^{2}.
\]

With \([U^{k}] := U_{R}^{k-1} - U_{L}^{k}\) we obtain

\[
f_{num,K}^{num}(U_{R}^{k-1}, U_{L}^{k}) [U^{k}] - \frac{1}{2} \left([U^{k}]\right)^{2}
\]

for the terms of the internal faces. As it was already described in [12], this quantity is non-negative for either the upwind or central numerical flux. It is

\[
f_{num,K}^{num}(U_{R}^{k-1}, U_{L}^{k}) [U^{k}] - \frac{1}{2} \left([U^{k}]\right)^{2} = \frac{1}{2} \left( [U^{k}] \right)^{2} \geq 0 \quad \text{with} \quad \left\{ \begin{array}{l} \sigma = 0 \text{ central flux,} \\ \sigma = 1 \text{ upwind flux,} \end{array} \right.
\]

and we get in total

\[
\frac{1}{2} \frac{d}{d t} \sum_{k=1}^{K} \frac{\Delta x_{k}}{2} \left\| u^{k}(t) \right\|_{N}^{2} = \frac{1}{2} \frac{d}{d t} \sum_{k=1}^{K} \frac{\Delta x_{k}}{2} \left\| U^{k}(t) \right\|_{N}^{2} - \sum_{k=1}^{K} u_{k}^{2} R_{R}^{T} B \left( f_{num,K}^{num} - \frac{1}{2} R_{L}^{k} u^{k} \right)
\]

\[
= \frac{1}{2} g(t)^{2} - \frac{1}{2} \left( U_{L}^{1}(t) - g(t) \right)^{2} - \frac{1}{2} \left( U_{R}^{K}(t) \right)^{2} - \frac{1}{2} \sum_{k=2}^{K} \left( [U^{k}(t)] \right)^{2}.
\]

The global norm is defined by

\[
\left\| U(t) \right\|_{N}^{2} := \sum_{k=1}^{K} \frac{\Delta x_{k}}{2} \left\| U^{k}(t) \right\|_{N}^{2},
\]

and \( U(0) \) is the interpolant of the initial condition \( u_{0} \) on the element. Integration from zero to \( T \) yields

\[
\left\| U(T) \right\|_{N}^{2} + \int_{0}^{T} \left( U_{R}^{K}(t) \right)^{2} d t + \int_{0}^{T} \left( U_{L}^{1}(t) - g(t) \right)^{2} d t + \sigma \int_{0}^{T} \sum_{k=2}^{K} \left( [U^{k}] \right)^{2} d t = \left\| U(0) \right\|_{N}^{2} + \int_{0}^{T} g^{2}(t) d t,
\]

which also satisfies

\[
\left\| U(T) \right\|_{N}^{2} + \int_{0}^{T} \left( U_{R}^{K}(t) \right)^{2} d t \leq \left\| U(0) \right\|_{N}^{2} + \int_{0}^{T} g^{2}(t) d t.
\]

In accordance to [12], equation (19) matches (9) except for the additional dissipation, which comes from the boundary conditions and the numerical fluxes, if an upwind flux is used. We see that in this formulation our SBP-CPR/FR is strongly stable and the energy at any time is bounded. This is not surprising, because in our formulation our method matches DGSEM from [12], where already this has been investigated for nodal Gauß-Lobatto points. However, we also consider a nodal Gauß-Legendre basis.

### 4.2 Error Equation

Now we analyse the time behaviour of the error. The error is given by \( E^{k} := u(x, \xi, t) - U^{k}(\xi, t) \). We split it also in two parts:

\[
E^{k} = \sum_{\xi = i_{P}}^{i_{P}+1} \left( \xi^{N}(u^{k}) - U^{k} \right) + \sum_{\xi = i_{P}^{p}}^{i_{P}^{p}} \left( u - \xi^{N}(u^{k}) \right).
\]
With the triangle inequality, we can bound this error by
\[ ||E^k||_N^2 \leq ||\varepsilon_p^k||_N^2 + ||\varepsilon_p^k||_N^2. \] (21)
\( \varepsilon_p^k \) is the interpolation error, which is the sum of the series truncation error and the aliasing error. As it was already described in [4, 6, 8, 17, 18], its continuous norms converge spectrally fast for the different bases under consideration. We denote by
\[ |u|_{H^{m,N}(-1,1)} := \left( \sum_{j=\min(m,N+1)}^m ||u^{(j)}||_{L^2(-1,1)}^2 \right)^{\frac{1}{2}} \]
the seminorms of the Sobolev space \( H^{m}(-1,1) \) and \( P_N \) is the projection operator of the truncated Legendre series. We get:
- Gauß-Lobatto/Gauß-Legendre points
  \[ ||u - \Pi(u)||_{L^2(-1,1)} \leq CN^{-m}|u|_{H^{m,N}(-1,1)}; \] (22)
- Legendre basis
  \[ ||u - P_N(u)||_{L^2(-1,1)} \leq CN^{-m}|u|_{H^{m,N}(-1,1)}; \] (23)
where \( C \) depends on \( m \). In view of our investigation, we need to consider our interpolation error not only in the standard interval \([-1, 1]\), but in each element \( e^k \). Therefore, we will transform our estimations (22)-(23) to every element. We get with the interval length \( \Delta x^k = x_R^k - x_L^k \):
- Gauß-Lobatto/Gauss-Legendre\(^3\) points (Combination of [6, Theorem 6.6.1] and [4, Section 5.4.4])
  \[ ||\varepsilon_p^k||_{H^n(e^k)} \leq C (\Delta x^k)^{n-\min(m,N)+\frac{1}{2}} N^{n-m+\frac{1}{2}} |u|_{H^{m,N}(e^k)}; \] (24)
for \( n = 0, 1 \). For Gauss-Lobatto, we can delete \( \frac{1}{2} \) on the right side of (24).
- Legendre basis ( [4, Section 5.4.4])
  \[ ||u - P_N^l(u)||_{H^n(e^k)} \leq C (\Delta x)^{n-\min(m,N)} N^{n-m} |u|_{H^{m,N}(e^k)}; \] (25)
for all \( 0 \leq n \leq l \) and \( P_N^l \) be the orthogonal projection of \( u \) onto \( P_N \), under the inner product of \( H^1(e^k) \).

We are considering a finite dimensional normed vector space. All norms are equivalent in this vector space and this allows us to bound the discrete norm in terms of the continuous ones. This implies that \( ||\varepsilon_p^k||_N \) in (21) decays spectrally fast in all cases of consideration. In other words, the part of the error we have to investigate in detail is \( \varepsilon_p^k \). This error describes the difference of the interpolation of \( u \) and the the spatial approximation \( U \). We derive now the error equation for \( \varepsilon_p^k \). \( u \) is the solution of the continuous equation
\[ \frac{\Delta x}{2} (\partial_t u, \varphi) + u \varphi \big|_{-1}^1 - (u, \partial_x \varphi) = 0, \] (26)
where \( (u, \varphi) := \int_{-1}^1 u \varphi \, d\xi \) defines the inner product. Equation (26) can be derived from the advection equation (8) by multiplication with the test function \( \varphi \), integrating over the standard element and integration-by-parts. With \( \varphi \in P_N \subset L^2 \) and \( u = \Pi(u) + \varepsilon_p^k \) we get for the continuous equation
\[ \frac{\Delta x_k}{2} \left( \partial_t \Pi(u)^k, \varphi^k \right) + \Pi(u)^k \varphi^k \big|_{-1}^1 - (\Pi(u)^k, \partial_x \varphi^k) = -\frac{\Delta x_k}{2} \left( \partial_t \varepsilon_p^k, \varphi^k \right) - \varepsilon_p^k \varphi^k \big|_{-1}^1 + (\varepsilon_p^k, \partial_x \varphi^k). \] (27)
\(^1\)If we use a modal Legendre basis in our CPR method and suppose exact integration. The interpolation operator in equation (20) can be replaced by the projection operator. The interpolation error. is only the series truncation error. The investigation can be carried out analogously. We will provide the estimations, but not consider this case here in detail.
\(^2\)See section 5.4.2 of [4] for detail.
\(^3\) A more detailed analysis can be found in [2, 3].
Remark 4.1. For Gauß-Lobatto nodes \( \varepsilon_p^k = 0 \) at the endpoints, since the interpolant is equal to the solution there. It is \( \varepsilon_p^k \big|_{-1} = 0 \).

Using integration-by-parts for \( \left( \varepsilon_p^k, \partial_x^k \right) \) yields
\[
\frac{\Delta x_k}{2} \left( \partial_t I^N(u)^k, \varphi^k \right) + \| I^N(u)^k \varphi^k \big|_{-1} - ( I^N(u)^k, \partial_x^k \varphi^k ) = - \frac{\Delta x_k}{2} \left( \partial_t \varepsilon_p^k, \varphi^k \right) - \left( \partial_x^k ( \varepsilon_p^k ), \varphi^k \right). \tag{28}
\]
Applying now interpolation and the discrete norm gives for the first term
\[
\left( \partial_t I^N(u)^k, \varphi^k \right) = \left( \partial_t I^N(u)^k, \varphi^k \right)_N + \left\{ \left( \partial_t \| I^N(u)^k \varphi^k \big|_{-1} - ( I^N(u)^k, \partial_x^k \varphi^k ) \right)_N \right\}. \tag{29}
\]
From [4, Section 5.43], we know that the integration error arising from the use of Gauß quadrature (Gauß-Legendre and Gauß-Lobatto) decays spectrally. It is for all \( \varphi \in P_N \) and \( m \geq 1 \)
\[
\left| (u, \varphi) - (u, \varphi)_N \right| \leq CN^{-m} |u|_{H^{m,N}(-1,1)} \| \varphi \|_{L^2(-1,1)},
\]
where \( C \) is a constant independent from \( m \) and \( u \). Since \( \varphi \in P_N \), we have for volume term in (28)
\[
\| I^N(u)^k \varphi^k \big|_{-1} = \frac{\partial_t I^N(u)^k, \varphi^k}{N} = \frac{\partial_t D M}{N} I^N(u)^k. \tag{30}
\]
Finally, the values of the interpolation polynomial at the boundaries of the element \((-1,1)\) can be approximated by a limitation process from the left side \( I^N(u)^k_{-1} \) and right side \( I^N(u)^k_{+1} \). To simplify the notation we define
\[
I^{num,k} \left( I^N(u)_k, I^N(u)_{k,+} \right) := \left( f^{num} \left( I^N(u)^{k-1}_R, I^N(u)_k \right), f^{num} \left( I^N(u)_k, I^N(u)^{k+1}_L \right) \right). \tag{31}
\]
We obtain for the approximation
\[
\varphi_k^N \big|_{-1} = \varphi_k^T \frac{R^T B M}{I^{num,k} \left( I^N(u)_k, I^N(u)_{k,+} \right)} + \left( \varphi_k^N \big|_{-1} - \varphi_k^T \frac{R^T B M}{I^{num,k} \left( I^N(u)_k, I^N(u)_{k,+} \right)} \right) \cdot \frac{\partial_t I^N(u)^k, \varphi^k}{N}, \tag{32}
\]
u is continuous \([m > 1] \). Using Gauß-Lobatto points the error term in the braces is zero, because the interpolation polynomial is evaluated at these boundaries and the numerical flux is unique. For Gauss-Legendre points, we get an additional error term, which corresponds to an interpolation error (in the pointwise sense) at these end points. The numerical flux is again unique and so the error term \( \varepsilon_2 \) reads
\[
\varepsilon_2^k := \left\{ \varphi_k^N \big|_{-1} - \varphi_k^T \frac{R^T B M}{I^{num,k} \left( I^N(u)_k, I^N(u)_{k,+} \right)} \right\},
\]
in every element \( k \). Finally, we obtain
\[
\frac{\Delta x_k}{2} \left( \partial_t I^N(u)^k, \varphi^k \right)_N + \varphi_k^T \frac{R^T B M}{I^{num,k} \left( I^N(u)_k, I^N(u)_{k,+} \right)} - \left( \partial_t \varphi^k \right)_N = - \frac{\Delta x_k}{2} \left\{ \left( \partial_t I^N(u)^k, \varphi^k \right) - \left( \partial_t \| I^N(u)^k \varphi^k \big|_{-1} - ( I^N(u)^k, \partial_x^k \varphi^k ) \right)_N \right\}
\]
\[
= - \varepsilon_2^k.
\]
Putting with the linearity of the numerical flux an equation for the error \( \varepsilon \), since for Gauß-Lobatto nodes, this error term is zero. We subtract (14) from (33) and get
\[
\frac{\Delta t_k}{2} \left( \partial_t \|N(u)^k\|_2, \varphi^k \right)_N = \frac{\Delta t_k}{2} \int_{\mathbb{T}^k} \rho^{\text{num},k} \left( \|N(u)^k\|_2 - \|N(u)^k\|_2 \right) \partial_t \|N(u)^k\|_2 \partial_t \varphi^k \right)_N,
\]
yields in (32)
\[
\frac{\Delta t_k}{2} \left( \partial_t \|N(u)^k\|_2, \varphi^k \right)_N + \varphi^k T_k B \int_{\mathbb{T}^k} \rho^{\text{num},k} \left( \|N(u)^k\|_2 - \|N(u)^k\|_2 \right) \partial_t \|N(u)^k\|_2 \partial_t \varphi^k \right)_N
= \frac{\Delta t_k}{2} \left( T_k(u), \varphi^k \right) + \frac{\Delta t_k}{2} \left( Q(u)^k, \varphi^k \right)_N - \varepsilon^k,
\]
with
\[
T_k(u) = - \left\{ \partial_t \varepsilon^k_p + \partial_t \varepsilon^k_p + Q(u)^k \right\}.
\]
\( Q \) measures the projection error of a polynomial of degree \( N \) to a polynomial of degree \( N - 1 \). Since \( u \) is bounded, also this value has to be bounded. Since (24), the interpolation error \( \varepsilon^k_p \) converges in \( N \) to zero, if \( m > 1 \) and the Sobolev norm of the solution is uniformly bounded in time\(^4\). For the time derivative, we get the boundedness of the norm by the relation \( \partial_t u = -\partial_t u \). \( \varepsilon^k_p \) is also bounded, because \( u \) is bounded and also continuous. For the numerical fluxes, this value describes the error between the interpolation polynomial at \( -1 \) and \( 1 \) and the numerical approximation by the numerical flux function at these boundaries. From a different perspective, this value can also be interpreted as the additional dissipation, which is added in the Gauß-Legendre case, since for Gauß-Lobatto nodes this error term is zero. We subtract (14) from (33) and get with the linearity of the numerical flux an equation for the error \( \varepsilon^k_1 = \|N(u)^k\|_2 - U^k \). It is
\[
\frac{\Delta t_k}{4} \left( \partial_t \|N(u)^k\|_2, \varphi^k \right)_N + \varphi^k T_k B \int_{\mathbb{T}^k} \rho^{\text{num},k} \left( \|N(u)^k\|_2 - \|N(u)^k\|_2 \right) \partial_t \|N(u)^k\|_2 \partial_t \varphi^k \right)_N
= \frac{\Delta t_k}{2} \left( T_k(u), \varphi^k \right) + \frac{\Delta t_k}{2} \left( Q(u)^k, \varphi^k \right)_N - \varepsilon^k,
\]
Putting \( \varphi^k = \varepsilon^k_1 \), we obtain the energy equation
\[
\frac{\Delta t_k}{4} \left( \varepsilon^k_1 \right)_N^2 + \frac{\Delta t_k}{2} \int_{\mathbb{T}^k} \rho^{\text{num},k} \left( \varepsilon^k_1 \right)_N \cdot \partial_t \varepsilon^k_1 \right)_N
= \frac{\Delta t_k}{2} \left( T_k(u), \varepsilon^k_1 \right) + \frac{\Delta t_k}{2} \left( Q(u)^k, \varepsilon^k_1 \right)_N - \varepsilon^k,
\]
with \( \varepsilon^k_2 = \left( \varepsilon^k_1 \right)_N^1 - \varepsilon^k_1 T_k B \int_{\mathbb{T}^k} \rho^{\text{num},k} \left( \|N(u)^k\|_2 - \|N(u)^k\|_2 \right) \partial_t \|N(u)^k\|_2 \partial_t \varphi^k \right)_N \). Summation-by-parts yields for \( \varepsilon^k_1 \)
\[
\left( \varepsilon^k_1, \partial_t \varepsilon^k_1 \right)_N = \varepsilon^k_1 T_k B \int_{\mathbb{T}^k} \rho^{\text{num},k} \left( \|N(u)^k\|_2 - \|N(u)^k\|_2 \right) \partial_t \|N(u)^k\|_2 \partial_t \varphi^k \right)_N - \varepsilon^k_1 T_k B \int_{\mathbb{T}^k} \rho^{\text{num},k} \left( \|N(u)^k\|_2 - \|N(u)^k\|_2 \right) \partial_t \|N(u)^k\|_2 \partial_t \varphi^k \right)_N,
\]
\(^4\) Therefore, we need the initial and boundary conditions in the model problem (8).
\[ \Leftrightarrow \quad \varepsilon_1^{T,k} M D \varepsilon_1^k = \frac{1}{2} \varepsilon_1^{T,k} R^T B R \varepsilon_1^k, \]

and we get

\[ \frac{\Delta x_k}{4} \frac{d}{dt} \| \varepsilon_1^k \|_N^2 + \varepsilon_1^{T,k} R^T B \left( f^\text{num,k} \left( (\varepsilon_1^k)^-, (\varepsilon_1^k)^+ \right) - \frac{1}{2} R \varepsilon_1^k \right) \]

\[ = \frac{\Delta x_k}{2} \left( T^k(u), \varepsilon_1^k \right) + \frac{\Delta x_k}{2} \left( Q(u)^k, \xi_1^k \right)_N - \varepsilon_2^k. \]

We sum over all elements and get

\[ \frac{1}{2} \frac{d}{dt} \sum_{k=1}^{K} \frac{\Delta x_k}{2} \| \varepsilon_1^k \|_N^2 + \sum_{k=1}^{K} \xi_1^{T,k} R^T B \left( f^\text{num,k} \left( (\varepsilon_1^k)^-, (\varepsilon_1^k)^+ \right) - \frac{1}{2} R \varepsilon_1^k \right) \]

\[ = \sum_{k=1}^{K} \left( \frac{\Delta x_k}{2} \left( T^k(u), \varepsilon_1^k \right) + \frac{\Delta x_k}{2} \left( Q(u)^k, \xi_1^k \right)_N - \varepsilon_2^k \right). \quad (34) \]

We define the numerical flux of the error by \( \varepsilon_1^\text{num,k} := f^\text{num,k} \left( (\varepsilon_1^k)^-, (\varepsilon_1^k)^+ \right) \) and get for the global energy of the error

\[ \frac{1}{2} \frac{d}{dt} \sum_{k=1}^{K} \frac{\Delta x_k}{2} \| \varepsilon_1^k \|_N^2 + \sum_{k=1}^{K} \xi_1^{T,k} R^T B \left( \varepsilon_1^\text{num,k} - \frac{1}{2} R \varepsilon_1^k \right) \]

\[ = \sum_{k=1}^{K} \left( \frac{\Delta x_k}{2} \left( T^k(u), \varepsilon_1^k \right) + \left( Q(u)^k, \xi_1^k \right)_N - \varepsilon_2^k \right). \quad (35) \]

This equation has the same form as (16) except the right side. We estimate the bracket on the right hand side by the Cauchy-Schwarz inequality. It is

\[ R = \sum_{k=1}^{K} \frac{\Delta x_k}{2} \left( T^k(u), \varepsilon_1^k \right) + \left( Q(u)^k, \xi_1^k \right)_N \]

\[ \leq \sqrt{\sum_{k=1}^{K} \frac{\Delta x_k}{2} \| T^k(u) \|^2} \sqrt{\sum_{k=1}^{K} \frac{\Delta x_k}{2} \| \varepsilon_1^k \|_N^2} + \sqrt{\sum_{k=1}^{K} \frac{\Delta x_k}{2} \| Q(u)^k \|_N^2} \sqrt{\sum_{k=1}^{K} \frac{\Delta x_k}{2} \| \xi_1^k \|_N^2}. \]

With the global norm over all elements and the equivalence between the continuous and discrete norms, we obtain

\[ R \leq \| \varepsilon_1 \|_T + \| Q \|_N \| \varepsilon_1 \|_N \equiv \tilde{E}(t) \| \varepsilon_1 \|_N \quad (36) \]

Now, we focus on \( \varepsilon_2^k = \left( \varepsilon_1^k |^\text{N}(u)^k \right|_{-1}^1 - \varepsilon_1^{T,k} R^T B f^\text{num,k} \left( |\text{N}(u)^k\cdot, |\text{N}(u)^{k,+}\right) \) with \( \varepsilon_1^k = 1^k(u) - U^k \in \mathbb{R}^N \).

\( \varepsilon_1^k \) are continuous and also uniformly continuous in the compact interval \([-1, 1]\) for all \( k \in \{1, \cdots, K\} \). \( R \)

is the restriction to the boundaries and \( |\text{N}(u)^k \in \mathbb{R}^N \). The first term can be expressed by \( \varepsilon_1^k |^\text{N}(u)^{k,-} \}_{-1}^1 = \varepsilon_1^{T,k} R^T B f^\text{num,k} \left( |\text{N}(u)^{k,-}, |\text{N}(u)^{k,+}\right) \), since \( \varepsilon_1^{T,k} R^T \) approximates /interpolates the value at the boundaries -1 and 1 exactly and \( \varepsilon_1^k \) are uniformly continuous. We may write

\[ \left| \varepsilon_2^k \right| = \left| \varepsilon_1^k |^\text{N}(u)^k \right|_{-1}^1 - \varepsilon_1^{T,k} R^T B f^\text{num,k} \left( |\text{N}(u)^{k,-}, |\text{N}(u)^{k,+}\right) \]

\[ = \left| \varepsilon_1^{T,k} R^T B |\text{N}(u)^k \right| - \varepsilon_1^{T,k} R^T B f^\text{num,k} \left( |\text{N}(u)^{k,-}, |\text{N}(u)^{k,+}\right) \]
\( \|u\|_{H^m} \) is uniformly bounded in time, \( I^N(u)^k \) is bounded in every element and so also the numerical flux \( f^{\text{num}} \). Therefore, the bracket is bounded.

The absolute value of \( \varepsilon_1^k \) can be estimate by twice the maximum value and since all norms are equivalent, we are able to estimate every \( \varepsilon_1^k \) by the global discrete norm. We obtain

\[
\sum_{k=1}^{K} |\varepsilon_1^k| \leq |\hat{M}(t)||\varepsilon_1||_N
\]

and altogether

\[
R - \sum_{k=1}^{K} \varepsilon_1^k \leq (\hat{E}(t) + |\hat{M}(t)|) \|\varepsilon_1\|_N = \hat{E}(t)||\varepsilon_1||_N. \tag{37}
\]

**Remark 4.2.** The estimations for \( \varepsilon_2^k \) are rough. As we already mentioned before, the error can be seen as an additional dissipation term, which is added in the Gauß-Legendre case. Therefore, a more accurate investigation of this term would be desirable, especially if not a linear problem is analysed, but this is still work in progress. Simultaneously, with inequality (35) and the assumption \( 0 < \sum_{k=1}^{K} \varepsilon_2^k \), the Gauß-Legendre error bounds should be less than in the Gauß-Lobatto case. Indeed, our numerical tests demonstrate that in all simulations using the Gauß-Legendre nodes lead to more accurate solutions.

Using estimation (37) in (35), we get an inequality for the global error equation of the total energy. It is

\[
\frac{1}{2} \frac{d}{dt} ||\varepsilon_1||_N^2 + \sum_{k=1}^{K} \varepsilon_1^T \hat{R} T \left( f^{\text{num}}(\varepsilon_1^k) - \frac{1}{2} \hat{R} \varepsilon_1^k \right) \leq \hat{E}(t)||\varepsilon_1||_N. \tag{38}
\]

Applying the same approach like in [12] and split the sum into three parts (One for the left physical boundary, one for the right physical boundary and a sum over the internal element endpoints), we get

\[
\sum_{k=1}^{K} \varepsilon_1^T \hat{R} T \left( f^{\text{num}}(\varepsilon_1^k) - \frac{1}{2} \hat{R} \varepsilon_1^k \right) = \sum_{k=1}^{K} \varepsilon_1^T \hat{R} T \left( f^{\text{num}}(\varepsilon_1^k) \right) = -E_L^k \left( f^{\text{num}}_{L}(1) - \frac{1}{2} E_L^k \right) + \sum_{k=2}^{K} f^{\text{num}}_{L}(k-1) \left( E_R^{k-1} + E_L^k \right) \left( E_R^{k-1} - E_L^k \right)
\]

\[
+ E_R^K \left( f^{\text{num}}_{R}(K) - \frac{1}{2} E_R^K \right).
\]

We describe with \( E_i \) \( (i = L, R) \) the approximated error \( \varepsilon_1 \), the indices give the position in the elements, \( f^{\text{num}}_{L}(k-1) \), \( E_L^k \) and \( f^{\text{num}}_{R}(K) \), the external states for the physical boundary contributions are zero, because \( I^N(u)^1 = g \) at the left boundary and the external state for \( U^1 \) is set to \( g \). At the right boundary, where the upwind numerical flux is used, it doesn’t matter what we set for the external state, since its coefficient in the numerical solution is zero. We get for the inner element with \( [E]^k = E_R^{k-1} - E_L^k \) similar to (17)

\[
\sum_{k=2}^{K} \left( f^{\text{num}}_{L}(k-1) - \frac{1}{2} \left( E_R^{k-1} + E_L^k \right) \right) \left( E_R^{k-1} - E_L^k \right) = \sum_{k=2}^{K} \frac{\sigma}{2} \left( [E]^k \right)^2 \geq 0,
\]

with \( \sigma = 0 \) central flux, \( \sigma = 1 \) upwind flux.
For the left and right boundaries, we get finally
\[
\begin{align*}
\text{left:} & \quad - E_L^1 \left( f_L \frac{\text{num}}{2} - \frac{1}{2} E_L^1 \right) = - E_L^1 \left( \frac{0 + E_L^1}{2} - \sigma \frac{E_L^1}{2} \right) = \frac{\sigma}{2} \left( E_L^1 \right)^2, \\
\text{right:} & \quad E_R^K \left( f_R \frac{\text{num}}{K} - \frac{1}{2} E_R^K \right) = E_R^K \left( \frac{0 + E_R^K}{2} + \frac{1}{2} \sigma E_R^K \right) = \frac{\sigma}{2} \left( E_R^K \right)^2.
\end{align*}
\]

Therefore, the energy growth rate is bounded by
\[
\frac{1}{2} \frac{d}{dt} \| \varepsilon_1 \|_N^2 + \frac{\sigma}{2} \left( \left( E_R^K \right)^2 + \left( E_L^1 \right)^2 \right) \leq E(t) \| \varepsilon_1 \|_N. \tag{39}
\]

It is \(BTs \geq 0\). This is exactly the same form as in [12] and we may estimate/bound similar to [12,14] the error in time. In the term \(E\), we have an additional term \(\tilde{M}\), but this has no influence in the estimation here. We now rewrite (39) in the following form:
\[
\frac{\partial}{\partial t} \| \varepsilon_1 \|_N + \frac{BTs}{\eta(t)} \| \varepsilon_1 \|_N \leq E(t). \tag{40}
\]

Like it was described in [14], we assume that the mean value of \(\eta(t)\) over any finite time interval is bounded by a positive constant \(\delta_0\) from below. This means that \(\eta \geq \delta_0 > 0\). Under the assumption for \(u\) also the right hand side \(E(t)\) is also bounded in time and we can put \(\max_{s \in [0,\infty)} E(s) \leq C_1 < \infty\). Applying these facts in (40), we integrate over the time and get the following inequality for the error
\[
\| \varepsilon_1(t) \|_N \leq \frac{1 - \exp(-\delta_0 t)}{\delta_0} C_1, \tag{41}
\]
see [14, Lemma 2.3] for details. We transferred the results from [12,14] to the SBP-CPR/FR framework and extended the basis also to Gauß-Legendre.

If the truncation error is bounded, the dissipative boundary conditions keep also the error bounded in time for both Gauß-Lobatto as well as Gauß-Legendre nodes. The selections of basis and numerical fluxes have an essential influence on the error behaviour.

It is clear that this approach can be easily transformed to multidimensional problems using a tensor product structure on structure grids. In the next section, we focus on the one parameter family of Vincent et al. [23,24] and transfer our analysis to this family and so to more general FR schemes. This generalisation evolve a more complex notation.

5 Error bounds for the parameter family of Vincent et al.

In our stability investigation in section 4.1, we used for the correction term \(C = M^{-1} R^T B\), which corresponds to the DGSEM from [11]. Here, we are now considering a more general correction term \(C\). In [21], the authors already studied stability in this context. We shortly repeat the main aspects and apply this approach to our case. Before equation (12), \(\varphi^{k,T} M\) was multiplied to (7). Instead of using \(M\) as the discrete norm, we may apply \(M + K\) analogues to [21]. The matrix \(M\) is associated as usual with the quadrature rule given by the polynomial basis and \(K\) is a symmetric matrix satisfying \(M + K > 0\), i.e. positive definite.
We are studying the change of the discrete norm \( \| u^k \|_{M+K}^2 = u^{k,T}(M + K)u^k \) for the total energy. Equation (12) reads
\[
\frac{\Delta x_k}{2} \varphi^{k,T} (M + K) \partial_t u^k = -\varphi^{k,T} (M + K) D u^k - \varphi^{k,T} (M + K) C (f_{\text{num},k} - R u^k) \tag{42}
\]
and with \( \varphi^k = u^k \):
\[
\frac{\Delta x_k}{2} u^{k,T} (M + K) \partial_t u^k = -u^{k,T} (M + K) D u^k - u^{k,T} (M + K) C (f_{\text{num},k} - R u^k). \tag{43}
\]

With the SBP property (5) the above equation (43) can be written as
\[
\frac{\Delta x_k}{2} u^{k,T} (M + K) \partial_t u^k
= -u^{k,T} (M + K) D u^k + u^{k,T} D^T M u^k - u^{k,T} D^T B R u^k
- u^{k,T} (M + K) C (f_{\text{num},k} - R u^k). \tag{44}
\]

Adding the equations (43) and (44) yields
\[
\frac{\Delta x_k}{4} \frac{d}{dt} \| u^k \|_{M+K}^2
= -u^{k,T} R^T B R u^k - \frac{1}{2} u^{k,T} R^T B R u^k. \tag{45}
\]
If \( KD \) is antisymmetric, then the first term on the right side of (45) is zero and we obtain with the correction term \( C := (M + K)^{-1} R^T B \) in (45)
\[
\frac{\Delta x_k}{4} \frac{d}{dt} \| u^k \|_{M+K}^2
= -u^{k,T} R^T B \left( f_{\text{num},k} - \frac{1}{2} R u^k \right). \tag{46}
\]

Equation (46) is analogous to equation (15), only the norm is different. We sum up over all elements and get finally equivalent to (16):
\[
\frac{1}{2} \frac{d}{dt} \sum_{k=1}^K \frac{\Delta x_k}{2} \| u^k \|_{M+K}^2 = -\sum_{k=1}^K u^{k,T} R^T B \left( f_{\text{num},k} - \frac{1}{2} R u^k \right). \tag{47}
\]

We follow the steps from section 4.1 in the study of the right side of (47). This yields a similar equation to (19):
\[
\| U(T) \|_K^2 + \int_0^T \left( U_K^k(t) \right)^2 \, dt \leq \| U(0) \|_K^2 + \int_0^T g^2(t) \, dt, \tag{48}
\]
where \( \| U(t) \|_K^2 := \sum_{k=1}^K \frac{\Delta x_k}{2} \| U^k(t) \|_{M+K}^2 \) is another global norm depending on \( K \).

Under the conditions on \( K \) and the correction terms \( C \) our methods are linear stable. This was already proven in [21, Theorem 5], together with the fact, that our methods correspond to a FR scheme, see [21] for details. For the one-parameter family of Vincent et al., the setting is
\[
K D = 0, \quad \text{where} \quad K = \kappa (D^N)^T M D^N, \tag{49}
\]

\( ^5 \)The results for the multi parameter family are similar to those about the one parameter family, since the one parameter family is contained in the extended range of schemes. In our investigation we only consider the one parameter family for simplicity.
and the selection of \( \kappa \) yields different numerical methods \([21]\).

For the discretion of the error, we follow the steps from subsection 4.2 and we evolve the error equation / estimation regarding the global norm \( || \cdot ||_K \). The error is given by \( E^K = u(x, \xi, t) - U^K(\xi, t) \) and it is

\[
||E^K||^2_K \leq ||\varepsilon_1^K||^2_K + ||\varepsilon_\nu^K||^2_K.
\] (50)

By the same argumentation as in subsection 4.2, the interpolation error \( ||\varepsilon_\nu^K||^2_K \) in (50) decays spectral fast due to the fact that all norms are equivalent in a finite dimensional normed vector space and we may apply estimations (22) - (25).

The key in our investigation is again to consider the error equation in terms of \( \varepsilon_1^k \). By the same Galerkin approach as in subsection 4.2, we derive again from the model equation (8) equation (27). However, we employ the discrete norm with respect to \( K \). Equation (29) reads

\[
\left( \partial_t \Pi^N(u)^k, \varphi^k \right) \left|_{M+K} \right. + \left\{ \left( \partial_t \Pi^N(u)^k, \varphi^k \right) - \left( \partial_t \Pi^N(u)^k, \varphi^k \right) \right|_{M+K},
\] (51)

but with assumption (49), we get for equation (30) the following

\[
\left( \Pi^N(u)^k, \partial_t \varphi^k \right) = \left( \Pi^N(u)^k, \partial_t \varphi^k \right)_{M+K} = \varphi^k D^T(M + K) \Pi^N(u)^k
\] (52)

Following the steps from (4.2), we get similar to (32)

\[
\frac{\Delta x_k}{2} \left( \partial_t \Pi^N(u)^k, \varphi^k \right)_{M+K} + \varphi^k D^T \left( M + K \right) \Pi^N(u)^k \left( \Pi^N(u)^k, \varphi^k \right)_{M+K} = \frac{\Delta x_k}{2} \left( \partial_t \phi^k + \varepsilon_\nu^k, \varphi^k \right) - \Delta x_k \left( \partial_t \phi^k + \varepsilon_\nu^k, \varphi^k \right)_{M+K} - \varepsilon_2^k.
\] (53)

Using

\[
\left( \partial_t \Pi^N(u)^k, \varphi^k \right) - \left( \partial_t \Pi^N(u)^k, \varphi^k \right)_{M+K} = \left( Q(u)^k, \varphi^k \right) - \left( Q(u)^k, \varphi^k \right)_{M+K},
\] yields in (53)

\[
\frac{\Delta x_k}{2} \left( \partial_t \Pi^N(u)^k, \varphi^k \right)_{M+K} + \varphi^k D^T \left( M + K \right) \Pi^N(u)^k \left( \Pi^N(u)^k, \varphi^k \right)_{M+K} - \varepsilon_2^k = \frac{\Delta x_k}{2} \left( T^k(u), \varphi^k \right)_{M+K} + \frac{\Delta x_k}{2} \left( Q(u)^k, \varphi^k \right)_{M+K} - \varepsilon_2^k.
\] (54)

Following again the steps from (4.2), we subtract (42) from (54). Before that we applied the SBP property (5), \( C = \left( M + K \right)^{-1} R^T B \) and the assumption (49) to equation (42). We obtain

\[
\frac{\Delta x_k}{2} \left( \partial_t \left( \Pi^N(u)^k - T^k \right), \varphi^k \right)_{M+K} + \varphi^k D^T \left( M + K \right) \Pi^N(u)^k \left( \Pi^N(u)^k - T^k \right) + \left( \Pi^N(u)^k - T^k \right) + \left( \Pi^N(u)^k - T^k \right)_{M+K} - \varepsilon_2^k,
\]

\[
\leftrightarrow \frac{\Delta x_k}{2} \left( \partial_t \phi^k, \varphi^k \right)_{M+K} + \varphi^k D^T \left( M + K \right) \Pi^N(u)^k \left( \Pi^N(u)^k, \varphi^k \right)_{M+K} - \varepsilon_2^k.
\]
where we also can write for the term \( \left( \varphi^k, \partial \xi \varphi^k \right)_{M+K} \) = \( \left( \varphi^k, \partial \xi \varphi^k \right)_{N} \), because the assumption (49) is fulfilled.

Putting \( \varphi^k = \varepsilon^k_1 \) and adding over all elements, we get a similar equation to (35):

\[
\frac{1}{2} \frac{d}{dt} \sum_{k=1}^{K} \frac{\Delta x_k}{2} ||\varepsilon^k_1||^2_{M+K} + \sum_{k=1}^{K} \xi_1^{T,k} \frac{R^T B}{2} \left( \xi_1^{\text{num},k} - \frac{1}{2} \frac{R \xi_1}{2} \right) = \sum_{k=1}^{K} \left( \left( T^k(u), \varepsilon^k_1 \right) + \left( Q(u)^k, \varepsilon^k_1 \right) \right)_{M+K} - \tilde{\varepsilon}^k_2. \tag{55}
\]

We may also estimate the right side of (55) and get analogous to (36), (37) and (38)

\[
\tilde{R} \leq \{ \varepsilon \|T\| + ||Q||_K \} \|\varepsilon_1\|_K \equiv E(t) ||\varepsilon_1||_K, \tag{56}
\]

\[
\tilde{R} - \sum_{k=1}^{N} \tilde{\varepsilon}_2^k \leq \left( \tilde{E}(t) + |\tilde{M}(t)| \right) ||\varepsilon_1||_K = \tilde{E}_V(t) ||\varepsilon_1||_K, \tag{57}
\]

and finally

\[
\frac{1}{2} \frac{d}{dt} ||\varepsilon_1||^2_K + \sum_{k=1}^{K} \xi_1^{T,k} \frac{R^T B}{2} \left( \xi_1^{\text{num},k} - \frac{1}{2} \frac{R \xi_1}{2} \right) \leq \tilde{E}_V(t) ||\varepsilon_1||_K. \tag{58}
\]

A comparison between (38) and (58) leads to the conclusion that the sum is equal and we can use the approach from before and we obtain

\[
\frac{\partial}{\partial t} ||\varepsilon_1||_K + \tilde{\eta}(t) ||\varepsilon_1||_K \leq \tilde{E}_V(t). \tag{59}
\]

and finally the following inequality for the error

\[
||\varepsilon_1(t)||_K \leq \frac{1 - \exp\left(-\tilde{\delta}_0 t\right)}{\tilde{\delta}_0} C_2 \tag{60}
\]

with \( \max_{s \in [0, \infty)} E_V(s) \leq C_2 \). We transferred the results from section 4.2 to the more general case of the one parameter family of Vincent et al. [23].

**Remark 5.1.** By the equivalence of the discrete norms, we may also estimate this error by the standard global norm (18), but we have to mention that this is only possible for the chosen parameter selection (grid, basis order \( N \)). If we increase for example the order \( N \) to infinity, also our constants in estimation of the norm equivalence will increase to infinity.

The discrete norms of the one parameter family correspond to some Sobolev norm. In our model problem (8) we need further assumptions about the regularity of our solution \( u \), if we consider \( u \) in the discrete norms with respect to \( K \).

### 6 Numerical Tests

In this section we consider numerical test, which demonstrate error bound (41). In [12, Subsection 6.2], the authors already consider a system for two space dimensions in the DGSEM framework and make equivalent observations like in the one-dimensional setting. Therefore, the more general case gives no further information. This is the reason, why we limit ourself to the one-dimensional case. However, we have a different topic in our investigation. We do not only apply Gauß-Lobatto nodes, but also employ a Gauß-Legendre basis. Our numerical simulations confirm our observation from remark 4.2 that the error term \( \tilde{\varepsilon}_2^k \) has an positive effect on the numerical scheme and we get more accurate solutions by using a Gauß-Legendre basis.
Also the influence of the different numerical fluxes is less important as in the Gauß-Lobatto case. We present several examples, which justify our observations, but we show also some limitations of our results. For the space discretisation, we chose the SBP-CPR/FR methods with $C = M^{-1} R^T B$. Results about the error behaviour of several other correction terms $\bar{C}$ can be found in [21] and are missed out in this section, because one does not obtain any further information from these simulations. We use an upwind flux (dotted lines) and central flux (lines) at the interior element interfaces$^6$. For time integration we use in all numerical examples a SSPRK(3,3), so that the time integration error is negligible and all elements are of uniform size.

![Graphs showing error behaviour as a function in time. The dashed lines are always the calculation with the upwind flux. Right side less elements than right. (c) and (d) early time behaviour.](image)

(a) $N = 4, K = 30, t = 20$
(b) $N = 4, K = 50, t = 20$
(c) $N = 4, K = 30, t = 4$
(d) $N = 4, K = 50, t = 4$

Figure 1: Error behaviour as a function in time. The dashed lines are always the calculation with the upwind flux. Right side less elements than right. (c) and (d) early time behaviour.

$^6$We apply always an upwind flux at the physical boundaries.
6.1 Error behaviour in One Space Dimension

Sine-Testcase

For comparison reason, we start our numerical section with the same example as in [12]. We analyse the error behaviour for $L = 2\pi$ and the initial condition $u_0 = \sin(12(x - 0.1))$, with the boundary condition $g(t)$ chosen so that the exact solution is $u(x, t) = \sin(12(x - t - 0.1))$. In Figure 1 we illustrate the discrete errors over time for different number of elements with a fourth order polynomial approximation. The errors are always bounded in time for all combinations (upwind/central flux and Gauß-Lobatto/Gauß-Legendre basis). We realise that the upwind flux errors reach in all cases its asymptotic values faster than the central flux errors. Simultaneously, the error bounds for the central flux are larger than for the upwind flux. These results are already formulated in [12] as some predictions (P1 and P2) and also the fact that the central flux errors are noiser than the upwind flux in all observations for all of the meshes and polynomial orders.

Here, we make the following two new observations:

First, the error bounds using Gauß-Lobatto points are larger than in the Gauß-Legendre case and secondly, the influence of the different numerical fluxes is less important than in the Gauß-Lobatto case. The first observation confirms our remark 4.2. The error term $e_2$ decreases the total error and we get a more accurate solution in this case. We interpret that as due to the fact that when using Gauß-Legendre nodes, which do not include the points at the element interfaces, additional dissipation comes form this and the influence of the dissipation from the upwind flux is less important compared to the Gauß-Lobatto case. If we increase the order of approximation, the error bounds of the different combinations should conform. Figure 2 justifies this consideration.

![Figure 2: Error behaviour as a function in time. The dashed lines are always the calculation with the upwind flux.](image)

(a) $N = 5$  (b) $N = 6$  (c) $N = 7$  (d) $N = 8$

Last, but not least, we study also the convergence speed and observe spectral accuracy in all cases, see Figure 3. This suggest that the approximation errors in $E(t)$ decay faster than $\frac{1}{K}$ grows, since with inequality (41) one predicts that the time asymptotic error is bounded by $E(t)/\delta_0$. This matches also with the investigation in [12].

Cosine-Testcase

As a second test case, we investigate the error behaviour for $L = 2\pi$ and the initial condition $u_0 = \cos(12(x - 0.1))$, with the boundary condition $g(t)$ chosen so that the exact solution is $u(x, t) = \cos(12(x - t - 0.1))$. With this test case we want to strengthen our results from before.

In Figure 4 we illustrate the discrete errors over time for different number of elements with a fourth and sixth order polynomial approximation. We make equivalent observations like before and see that using Gauß-Legendre nodes in our scheme yields to more accurate solutions as Gauß-Lobatto nodes. Also the difference between the upwind flux error and central flux error is not so large.
Figure 3: Convergences in time asymptotic errors (last value) as functions of $N$ for $K = 50$.

Figure 4: Error behaviour as a function in time with 50-elements.

(a) $N = 4$

(b) $N = 6$
6.2 Limitations and Counterexamples

We make a series of test calculations and most of the time, we make the above observations. Nevertheless, there are several examples, which qualify our results and also some predictions of [12]. We demonstrate and interpret these examples. In the error plot 3, one may realize that the upwind error lies beyond the central error if a polynomial approximation of order three is used. In fact, we see this clearly in Figure 5 (a). Here, the central error lies above the upwind error and also the asymptotic state is nearly the same\(^7\). We may interpret this that using polynomial order 3 in our schemes is to inaccurate for the approximation with the Gauss-Lobatto basis. Then, with applying an upwind flux yields too much dissipation into our calculation and we get this unpredictable behaviour, which damps our observation from above and also some predictions from [12]. In Figure 5 (b), we get a similar error behaviour as before, if we decrease the number of elements\(^8\) \(K\). With the higher jumps at the element interfaces the upwind flux yields to a more inexact numerical solution. We may conclude that we need an adequate number of elements to get the predicted results in the Gauss-Lobatto case. However, the numerical errors (upwind and central) with the Gauss-Legendre basis show the suspected behaviours from our results and one may interpret that as an advantage by using these type of basis, but this is a little bit misleading. In Figure 6 (a) we see the numerical errors of the cosine-case if we use polynomial order 3 and 20 elements. Here, the errors using Gauss-Lobatto basis show the expected manner and the errors with Gauss-Legendre basis do not. We assume that by using Gauss-Legendre basis and an upwind flux the jumps between the element interfaces is too high and we get this effect. If we again increase the number of elements and so the degrees of freedom, we realise a change in the error behaviours, compare 6 (a)-(c). The above leads to the conclusion that we need an adequate number of elements to get the expected behaviour.

Already in section 3 we mentioned to present an example, where the norm of solution \(\|u\|_{H^m}\) is not uniform bounded in time. We select our initial and boundary conditions in such way, that we get as the solution \(u(t, x) = (x - t)^8\). Figure 7 demonstrates the unbounded increase of the error for different times.

\(^7\)We assume that the noise state is periodic with the central flux

\(^8\)In [7] the influence of the dispersion and dissipation errors of Gauss-Legendre and Gauss-Lobatto is investigate also in respect to the number of elements.
Figure 6: Error behaviour as a function in time.

(a) $N = 3$, $K = 20$, $t = 20$
(b) $N = 3$, $K = 50$, $t = 20$
(c) $N = 3$, $K = 80$, $t = 20$

Figure 7: Error behaviour as a function in time.

(a) $N = 4$, $K = 50$, $t = 20$
(b) $N = 4$, $K = 50$, $t = 40$
7 Summary and Conclusion

In this paper, we transfer the results about the bounded error growth from the discontinuous Galerkin spectral element method [12] to the more general framework of SBP-CPR/FR methods. Furthermore, we extend the investigation by including Gauß-Legendre basis, where [12] considers only Gauß-Lobatto basis. Indeed, for both bases (Gauß-Lobatto/Gauß-Legendre), the numerical flux used at element boundaries affects the error growth behaviour. If an adequate number of elements is used, the upwind flux leads to better results. The asymptotic values are smaller and are reached in a shorter time period. At once, also the selection of basis has a big influence and in our opinion is even more important. Using Gauß-Legende basis, the error reaches the asymptotic value faster and to a lower value than by using Gauß-Lobatto basis in all simulations under consideration. Also, the impact of the different numerical fluxes (central/upwind) is by applying Gauß-Legendre basis less important as in the Gauß-Lobatto case, especially using a low order polynomial approximation. These effects decrease when the order of polynomial approximation is increased and/or using more elements (which also increase the resolution).

The investigation implies that the usage of Gauß-Legende basis has some advantages compared to Gauß-Lobatto and should be preferred. However, there are several points, which we have to mention before we conclude.

We investigate a trivial model problem (8), where the flux function is simple \( f(u) = u(x,t) \). Already by using the more complicated flux \( f(u) = a(x)u(x,t) \) several problems arise in the discretisation by using Gauß-Legende nodes, see [13] for details. The reason is that Gauß-Legende points do not include the boundary points in one element, and we get some aliasing effect if we are not careful in the discretisation. In [19], the author proves a way to solve these issues by applying further correction terms to approximate the boundary terms correctly. For non-linear flux functions stability problems rise automatically. The aliasing effect is quite stronger and to remedy these issues, further correction terms are needed [21]. By including the boundary points, these correction terms are simpler and better understand. [20] provides the correction terms for the shallow water equation using Gauß-Legende nodes, but to this point it is not clear how to construct these terms for more complicated equations like Euler-equation. In our calculations, the time integration analysis was negligible, but in real time simulation and practical use it is also an important issue. In [7], the authors already investigate the time-step restriction by using Gauß-Lobatto or Gauß-Legende nodes in the DGSEM and find out that Gauß-Lobatto nodes have favoured properties.

The above mentioned issues are not unimportant. However, for linear problems with adequate initial conditions the usage of Gauß-Legende basis should be taken into account. In our tests, the asymptotic error values are always reached faster and to a smaller amount. Nevertheless, further studies are necessary. First, one has to analyse the impact of \( \tilde{\varepsilon}_2 \) in detail. Secondly, what happens with the approximation error if jumps for the initial conditions or even more complex flux functions are considered?

References


