Substituting Compact Disks in Stable Planes

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Abstract

Given two stable planes of dimension 2\(l\) containing compact topological 2\(l\)-disks whose boundaries are isomorphic unitals, we show that it is possible to substitute one of these disks with the other, thus changing the line system in one of the planes, and that this produces another stable plane. The proof uses intersection theory in singular homology in order to show that existence of interior intersection points of two secants can be detected by looking at their traces in the unital. We conclude with a class of concrete examples based on P. Sperner’s 4-dimensional affine planes with group \(C^* \cdot SU_2\) \(C\).

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1 Introduction

Substituting compact disks is an established tool in the construction of 2-dimensional stable planes \([12]\); in fact, Hilbert’s first example of a non-desarguesian projective plane was obtained in this way, see \([3]\), §23. A 2-dimensional compact disk is just a convex topological disk whose boundary is an oval in the point set of a 2-dimensional stable plane. Here, convex means that each line intersects the disk in a connected set (possibly empty). Given two compact disks in two stable planes of dimension 2, it is possible to change the geometry of either plane by cutting out the given disk and fitting the disk from the other plane instead. The boundaries will always match, because their induced geometry is trivial. Moreover, it is not so hard to show that the construction produces another stable plane, essentially because intersecting secants of the disk can be recognized by the alternating positions of their boundary points \([12]\).

Our aim here is to make this construction tool available in stable planes of arbitrary (finite) dimension. The main obstacle to this construction was the problem of determining when two lines meet in the interior of the disk, just by looking at their traces on the boundary. We solve this problem using intersection numbers in the setting of algebraic topology, see \([1]\), VII.4. A similar problem occurred in \([6]\) and was solved using intersection theory in the differentiable setting.

There are restraints to be taken into account, however, as the geometry on the boundary will now be a unital, and in most cases, the two unitals will not match. At present
only one class of higher dimensional examples are known where the replacement can in
fact be done, but they are interesting and add to our insight into the relationship between
a projective plane and a unital contained in it. At least we can prove that the matching
of the boundary unitals is the only condition needed.

The more complicated structure of the boundary is also the reason why we do not (yet)
consider compact disks as geometries in their own right, independently of a surrounding
plane. In dimension two, this presented little difficulty [12].

2 Construction and Main Result

DEFINITION 2.1 A stable plane \((M, \mathcal{L})\) consists of a locally compact point space \(M\)
of positive (inductive) dimension and a set \(\mathcal{L}\) of subsets of \(M\), called lines, subject to the
following conditions. Two distinct points \(p, q \in M\) are joined by a unique line \(p \vee q\), and
with respect to a suitable topology on \(\mathcal{L}\), the map \(\vee\) is continuous. Moreover, the set of
intersecting pairs of lines is open (stability), and on this set, the intersection map \(\wedge\) is
continuous. For more information see, e.g., [5] or the survey [2].

We shall assume throughout that the point space has finite topological dimension. It
is known that then \(\dim M = 2l \in \{2, 4, 8, 16\}\), where \(l\) is the dimension of a line. The
following facts will be used frequently.

PROPOSITION 2.2 Let \((M, \mathcal{L})\) be a stable plane.

a) The incidence relation \(F = \{(p, L) \mid p \in L\}\) is closed in \(M \times \mathcal{L}\).

b) A set \(K \subseteq \mathcal{L}\) has compact closure in \(\mathcal{L}\) if, and only if, there is a compact set of
points meeting all lines \(L \in K\).

Proof. Assertion (a) is an easy consequence of the definition. For (b), see [5], 1.17.

DEFINITION 2.3 A hyperunital in a stable plane \((M, \mathcal{L})\) is a subset \(U \subseteq M\) homeo-
morphic to the sphere \(S_{2l-1}\), such that the intersection \(U \cap L\) with any line \(L\) is either
empty, or a singleton, or homeomorphic to \(S_{l-1}\). Accordingly, the line \(L\) is called a passing
line, a tangent, or a secant. Moreover, we require that each point \(u \in U\) lies on a unique
tangent \(T_u\). Often a unital is considered as an incidence geometry whose blocks are the
intersections \(U \cap L\) with all secants \(L\).

For simplicity, we shall just speak of unitals when we mean hyperunitals, although the
general notion of unitals allows for spheres of dimensions other than \(2l - 1\); compare [4].
The standard examples of (hyper-)unitals are the Euclidean spheres in the classical affine
planes \(\mathbb{F}^2, \mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}\); they consist of the absolute points of a hyperbolic polarity.
Other examples of unitals can be found in [13], [14].
DEFINITION 2.4 Suppose that in a stable plane \((M, \mathcal{L})\), we are given a compact subset \(D \subseteq M\) homeomorphic to the topological disk \(\mathbb{D}_2\), and that the boundary \(U = \partial D\) is a (hyper-)unital. Assume moreover that every secant meets \(D\) in a disk \(D \cap L \approx \mathbb{D}_1\) with boundary \(\partial(D \cap L) = U \cap L\). Then we call the induced geometry \((D, \mathcal{L}_D)\), whose line set \(\mathcal{L}_D\) consists of the intersections \(D \cap L\) of all secants \(L\), a compact disk geometry.

In [12], 2-dimensional compact disk geometries are simply called ‘compact disks’. We shall mostly follow this usage, but whenever we feel that there is some danger of confusion, we shall speak of topological compact disks or of compact disk geometries. Note also that Stroppel [12] does not require that the boundaries of his compact disks have a unique tangent at each point. He even allows line segments to be contained in the boundary.

We point out that the existence of a compact disk forces the lines of our plane to be \(l\)-manifolds. In general, this is not known to be true.

We are now ready to describe our construction. We suppose that we are given two stable planes \((M, \mathcal{L})\) and \((M', \mathcal{L}')\) of the same dimension \(2l\), containing compact disks \((D, \mathcal{L}_D)\) and \((D', \mathcal{L}'_D)\), respectively. We require that the unitals \(U = \partial D\) and \(U' = \partial D'\) are isomorphic. Every homeomorphism of the sphere \(\partial D\) extends to a homeomorphism of the cone \(D = C(\partial D)\), hence we can then find a homeomorphism \(\varphi : D' \to D\) inducing an isomorphism of the unitals. That is, for every secant \(L'\) of \(D'\), there is a secant \(L = \psi(L')\) of \(D\) such that

\[\varphi(U' \cap L') = U \cap \psi(L').\]

We extend the map \(\psi\) to the set of tangents by defining \(\psi(T_x) := T_{\varphi(x)}\).

We define a new plane \((M, \tilde{\mathcal{L}})\) on the point set \(M\) by taking as lines all lines \(L \in \mathcal{L}\) disjoint from the open disk \(\text{int} D = D \setminus U\) and the sets

\[\tilde{L} := (L \setminus D) \cup \varphi(L' \cap D'),\]

where \(L\) is a secant of \(D\) and \(L' \in \mathcal{L}'\) is the unique line such that \(\varphi(U' \cap L') = U \cap L\) (in other words, \(L' = \psi^{-1}(L)\)). Note that \(\tilde{L}\) is homeomorphic to \(L\). We define a topology on \(\tilde{\mathcal{L}}\) by insisting that the map \(L \mapsto \tilde{L}\) is a homeomorphism, where we put \(\tilde{L} = L\) for lines not meeting \(\text{int} D\).

Note that by construction, \(\varphi\) is an isomorphism of \((D', \mathcal{L}'_D)\) onto \((D, \mathcal{L}_D)\). The purpose of this paper is to prove the following.

**THEOREM 2.5** The incidence structure \((M, \tilde{\mathcal{L}})\) constructed above is a \(2l\)-dimensional stable plane.
3 Topology and Geometry of a Compact Disk

We assume generally that a stable plane containing a compact disk geometry $D$ is given. First, we show that a secant meets the boundary of the compact disk transversally in some weak sense.

**Lemma 3.1** Let $L$ be a secant of the unital $U = \partial D$. Then $U \cap L$ is the boundary of both $(\text{int } D) \cap L$ and $L \setminus D$.

*Proof.* By definition of a compact disk geometry, $U \cap L$ is the boundary of the disk $D \cap L$ and, hence, also of its interior. The sphere $U \cap L$ locally separates $L$, and a nonempty open disk neighbourhood of $x \in U \cap L$ is split into an open subset of $\text{int } D$ and one of $M \setminus D$. \hfill \square

**Lemma 3.2** Let $L_n$ be a sequence of lines converging to a secant $L$. Then every point $x \in U \cap L$ is the limit of some sequence $x_n \in U \cap L_n$.

*Proof.* Let $V$ be a connected neighbourhood of $x$ in $M$. We need to construct a point $x_n \in U \cap V \cap L_n$ for some $n$. To this end, consider an arc $A$ in $V \cap L$ whose endpoints $a, b$ are on different sides of $U$ (i.e., one is in $\text{int } D$ and the other, in $M \setminus D$). Project the arc into the line $L_n$ from some fixed center $c$. The projection map is given by $\pi_n(y) = (y \lor c) \cap L_n$. Using compactness of the arc together with continuity of join and intersection one obtains that the projected arc $A_n = \pi_n(A)$ will be contained in $V$ for large $n$. Indeed, if $y_n \in A_n \setminus V$, then $\pi_n^{-1}(y_n)$ and $y_n$ have a common accumulation point; this is a contradiction. Moreover, the end points $a_n, b_n$ of $A_n$ will be on distinct sides of $U$ for large $n$. Therefore, $A_n$ has to intersect $U$ in a point $x_n$, as desired. \hfill \square

Next we examine the map $\psi$ constructed in the previous section.

**Lemma 3.3** Let $\mathcal{L}_U$ be the set of lines meeting $U$, and likewise for $\mathcal{L}_U'$. The map $\psi : \mathcal{L}_U' \rightarrow \mathcal{L}_U$ is continuous (in fact, a homeomorphism).

*Proof.* Consider first a secant $L' \in \mathcal{L}'$ and a sequence $L'_n \rightarrow L'$. Choose two distinct points $x, y \in U' \cap L'$. By 3.2 there are sequences of points $x_n, y_n \in U' \cap L'_n$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$. Then $\psi(L'_n) = \psi(x_n \lor y_n) = \varphi(x_n) \lor \varphi(y_n)$ converges to $\varphi(x) \lor \varphi(y) = \psi(L')$.

Now let $L' = T_x$ be a tangent of $U'$. Again suppose that $L'_n \rightarrow L'$, where each line $L'_n$ meets $U'$. Then every sequence $x_n \in U' \cap L'_n$ converges to $x$. Since $\varphi(x_n) \rightarrow \varphi(x)$, the sequence $L_n = \psi(L'_n)$ is relatively compact, and accumulates only at elements of $\mathcal{L}_{\varphi(x)}$, see 2.2. If some subsequence would converge to a secant $S \in \mathcal{L}_{\varphi(x)}$, then every point of $U \cap S$ would be the limit of some sequence $\varphi(y_n)$ with $y_n \in U' \cap L_n$, again by 3.2. This contradicts the fact that $y_n \rightarrow x$. \hfill \square

**Theorem 3.4** Two modified lines $\tilde{L}_1, \tilde{L}_2$ meet in the interior $\text{int } D$ of the compact disk if, and only if, the original lines $L_1$ and $L_2$ do.
Proof. For the proof, we only need to look at the compact disk \( D \) itself. Hence we ignore the remainder of the plane and we suppress the map \( \varphi \). Moreover, we identify \( D \) with the standard disk \( \mathbb{D}_2 \). Thus we consider two line systems \( \mathcal{L} \) and \( \mathcal{L}' \) on the same topological disk \( D = \mathbb{D}_2 \) with boundary \( U = S_{2l-1} \). We consider lines \( L_i \in \mathcal{L} \) and \( L'_i \in \mathcal{L}' \) for \( i = 1, 2 \) such that \( U \cap L_i = U \cap L'_i =: A_i \). We assume that \( A_1 \cap A_2 = \emptyset \), and that \( L_1 \) intersects \( L_2 \) (note that lines are now subsets of \( D \)). We have to show that \( L'_1 \) intersects \( L'_2 \).

The proof rests on the easy observation that lines of a stable plane always intersect transversally, compare [5], 1.4. In fact, if lines are manifolds, then any intersection point has a neighbourhood homeomorphic to \( \mathbb{D}_l \times \mathbb{D}_l \), and the given lines correspond to the factors \( \mathbb{D}_l \times \{ 0 \} \) and \( \{ 0 \} \times \mathbb{D}_l \). This fact allows us to use intersection numbers as defined in [1], VII.4 as a tool in order to obtain our result. More precisely, we shall see that the two line pairs have the same intersection pairing, and that the intersection pairing is nonzero if, and only if, the lines have a common (interior) point. In what follows, homology will always be singular with integer coefficients.

We use the intersection pairing as defined in [1], VII.4.1:

\[
H_1(L_1, A_1) \times H_1(L_2, A_2) \xrightarrow{\times} H_2L_1 \times L_2, A_1 \times L_2 \cup L_1 \times A_2 \xrightarrow{d_*} H_2(\mathbb{R}^2 \setminus \{ 0 \});
\]

here, \( \times \) denotes the homology product, and the map \( d_* \) is induced by \( d(x, y) = x - y \). We have omitted a sign factor \((-1)^l \), because the interesting cases to us are those where \( l \) is even. We denote by \( \zeta \circ \eta \in H_2(\mathbb{R}^2 \setminus \{ 0 \}) \cong \mathbb{Z} \) the image of \((\zeta, \eta)\) under this pairing. If \( L_1 \cap L_2 = \emptyset \), then [1], VII.4.6 asserts that the pairing is identically zero.

Now assume that \( L_1 \cap L_2 = p \in \text{int} \, D \). We claim that the pairing is nonzero in this case. Let \( V \approx \mathbb{D}_l \times \mathbb{D}_l \) be a product neighbourhood of \( p \) as described above, and write \( V_i := V \cap L_i, L_i^x := L_i \setminus \{ p \}, V_i^x := V_i \setminus \{ p \} \). Naturality of the homology product ([1], VII.2.7) yields a commutative diagram with vertical arrows induced by inclusion,

\[
\begin{array}{ccc}
H_1(L_1, A_1) \times H_1(L_2, A_2) & \xrightarrow{\times} & H_2(L_1 \times L_2, A_1 \times L_2 \cup L_1 \times A_2) \\
\downarrow \cong & & \downarrow \\
H_1(L_1, L_1^x) \times H_1(L_2, L_2^x) & \xrightarrow{\times} & H_2(L_1 \times L_2, L_1^x \times L_2 \cup L_1 \times L_2^x) \\
\downarrow \cong & & \downarrow \\
H_1(V_1, V_1^x) \times H_1(V_2, V_2^x) & \xrightarrow{\times} & H_2(V_1 \times V_2, V_1^x \times V_2 \cup V_1 \times V_2^x)
\end{array}
\]

The arrows marked with \( \cong \) are isomorphisms; for the upper one, use the triple sequence and the fact that inclusion \( A_i \to L_i^x \) is a homotopy equivalence, and for the lower one, use excision. It follows from the diagram that up to isomorphism, the pairings in the top and bottom lines are the same. For the bottom line, [1], VII.4.10 asserts that the product \( \zeta \circ \eta \) of two generators is a generator of \( H_2(\mathbb{R}^2 \setminus \{ 0 \}) \).

Since we are working in the convex set \( D \), the inclusion maps \( L_i \to D \) and \( L'_i \to D \) are homotopic rel \( A_i \). This implies that the intersection pairings associated with \( (L_1, L_2) \) and with \( (L'_1, L'_2) \) are the same, up to isomorphism; we see this by rewriting the definition
of the pairing, as in [1], VII.4.14; namely, the pairing is given by the composite

\[ H_i(L_1, A_1) \times H_i(L_2, A_2) \xrightarrow{\times} H_2(L_1 \times L_2, A_1 \times L_2 \cup L_1 \times A_2) \]

\[ \xrightarrow{j^*} H_2(\mathbb{R}^{2i} \times \mathbb{R}^{2i}, \mathbb{R}^{2i} \times \mathbb{R}^{2i} \setminus \Delta) \xrightarrow{g_*^{-1}} H_2(\mathbb{R}^{2i}, \mathbb{R}^{2i} \setminus 0); \]

here, \( j \) is the inclusion map, \( \Delta \) denotes the diagonal in \( \mathbb{R}^{2i} \times \mathbb{R}^{2i} \), and \( g \) is the map \( (\mathbb{R}^{2i}, \mathbb{R}^{2i} \setminus 0) \to (\mathbb{R}^{2i} \times \mathbb{R}^{2i}, \mathbb{R}^{2i} \times \mathbb{R}^{2i} \setminus \Delta) \) defined by \( g(x) = (x, 0) \). Now in order to compare with the pairing for \((L_1', L_2')\), we use homeomorphisms \( f_i : L_i \to L_i' \) fixing every point of \( A_i \). Considered as maps into \( D \), these maps are homotopic rel \( A_i \) to the identity maps of \( L_i \). During the homotopy, \( A_1 \) remains fixed and may be kept disjoint from the image of \( L_2 \), and vice versa. This implies that the following diagram is commutative up to homotopy of pair maps.

\[ \begin{array}{ccc}
(L_1 \times L_2, A_1 \times L_2 \cup L_1 \times A_2) & \xrightarrow{j} & (\mathbb{R}^{2i} \times \mathbb{R}^{2i}, \mathbb{R}^{2i} \times \mathbb{R}^{2i} \setminus \Delta) \\
\downarrow{f_1 \times f_2} & & \downarrow{j'}
\end{array} \]

Applying homology and using naturality of homology products we obtain that the two pairings are essentially the same. \qed

4 Proof of Theorem 2.5

We divide the proof into a sequence of propositions.

PROPOSITION 4.1 Any two points in \( M \) are joined by a unique line \( \tilde{L} \in \mathcal{L} \).

Proof. By Theorem 3.4, two lines \( \tilde{L}_1, \tilde{L}_2 \) meeting in \( M \setminus D \) do not meet in \( \text{int} \, D \), because the original lines \( L_1, L_2 \) do not have a second common point. It is obvious that two points of \( M \setminus \text{int} \, D \) are joined by a unique line, and likewise for two points of \( D \). Thus, it suffices to show that a point \( x \in M \setminus D \) can be joined to every point in \( \text{int} \, D \) by some line. To this end, we consider the set \( X \) of all points in \( \text{int} \, D \) that can be joined to \( x \), and we show that \( X \) is nonempty and both open and closed in \( \text{int} \, D \). The line \( L \in \mathcal{L} \) joining \( x \) to any point in \( \text{int} \, D \) is a secant, hence \( \tilde{L} \) is also a secant, and \( X \) is indeed not empty.

The pencil \( \mathcal{X} := \tilde{\mathcal{L}}_x \approx \mathcal{L}_x \) is compact according to 2.2. For the same reason, the subset \( \mathcal{Y} \subseteq \mathcal{X} \) of all lines containing \( x \) and meeting \( U \) is also compact in the topologies of \( \tilde{\mathcal{C}} \) and of \( \mathcal{L} \). Using the homeomorphism \( \psi \), see 3.3, we obtain a compact set \( \psi^{-1}(\mathcal{Y}) \) of lines of
(\(M', \mathcal{L}'\)). It follows that the set \(X\) of all points in \(\text{int } D\) contained in an element of \(\mathcal{Y}\) is closed in \(\text{int } D\).

For the proof of openness of \(X\), we need the fact that \(\mathcal{X} \approx \mathcal{L}_x\) is an \(l\)-manifold (homeomorphic to \(S_2\)). The subset \(\mathcal{Z} \subseteq \mathcal{X}\) of all secants containing \(x\) is open, because \(\mathcal{L}\) is a secant if, and only if, \(L\) meets the open set \(\text{int } D\). It follows that \(\mathcal{W} := \psi^{-1}(\mathcal{Z})\) is an \(l\)-manifold of lines in the stable plane (\(\text{int } D', \mathcal{L}'_{\text{int } D'}\)) isomorphic to (\(\text{int } D, \hat{\mathcal{L}}_{\text{int } D}\)).

We know that the lines in \(\mathcal{W}\) are pairwise disjoint, and this implies that their union \(W\) is a 2\(l\)-manifold (hence open in \(\text{int } D'\) by domain invariance). Indeed, the line \(\hat{L}_p \in \mathcal{W}\) containing a point \(p \in W\) depends continuously on \(p\) by 2.2 and because \(\mathcal{W} \subseteq \mathcal{L}'_{\text{int } D}\) is closed and its elements are pairwise disjoint. Sending \(p \in W\) to the pair \((\hat{L}_p, p \lor q)\) with \(q \in \text{int } D' \setminus \hat{L}_p\) thus furnishes a local coordinate system for a 2\(l\)-manifold.

\[\square\]

**Proposition 4.2** The join operation is continuous in \((M, \hat{\mathcal{L}})\).

**Proof.** Let two convergent sequences \(p_n \to p\) and \(q_n \to q\) in \(M\) be given, \(p \neq q\). We have to show that \(\hat{L}_n := p_n \lor q_n \to p \lor q\). We may assume that all points \(p_n\) and \(p\) belong to \(D\), or that these points all belong to \(M \setminus \text{int } D\), and similarly for \(q_n\) and \(q\). No proof is required if both sequences and their limits belong to the same set, \(D\) or \(M \setminus \text{int } D\). So let us assume that \(p_n\) and \(p\) belong to \(D\) and \(q_n, q\) do not belong to \(\text{int } D\). Let \(L_n\) and \(\psi^{-1}(L_n) = L'_n\) be the lines used in the construction of \(\hat{L}_n\). We consider the original stable plane \((M, \mathcal{L})\), and we see that the lines \(L_n\) all meet the compact point set \(\{q_n \mid n \in \mathbb{N}\} \cup \{q\}\), so that the closure of this set of lines in \(\mathcal{L}\) is compact by 2.2. Using the homeomorphism \(\psi : \mathcal{L}'_{U'} \to \mathcal{L}_U\) from 3.3, we see that the same holds for the set of lines \(L'_n\) in \(\mathcal{L}'\). Thus, every subsequence of either sequence of lines contains a convergent subsequence. The limit of such a subsequence is a line containing \(q\) or \(p\), respectively, because the graphs of the incidence relations of the original stable planes are closed. We conclude that the given sequence \(\hat{L}_n\) converges to \(p \lor q\).

\[\square\]

The next proposition completes the proof of Theorem 2.5.

**Proposition 4.3** The plane \((M, \hat{\mathcal{L}})\) satisfies the stability axiom, and intersection is continuous.

**Proof.** Given two intersecting lines \(\hat{L}_1\) and \(\hat{L}_2\), we have to show that every line pair sufficiently close to \((\hat{L}_1, \hat{L}_2)\) intersects in a point close to \(x := \hat{L}_1 \land \hat{L}_2\). This is true if \(x \notin U\). If \(x \in U\), and \((\hat{K}_1, \hat{K}_2)\) is sufficiently close to \((\hat{L}_1, \hat{L}_2)\), then the original lines \(K_1\) and \(K_2\) intersect, and this carries over to the modified lines \(\hat{K}_1\) and \(\hat{K}_2\) by Theorem 3.4. Intersection points in \(M \setminus \text{int } D\) will be close to \(x\), since \((M, \mathcal{L})\) is a stable plane. The same is true for intersection points in \(D\), because the map \(\psi\) is continuous by 3.3 and \((M', \mathcal{L}')\) is a stable plane.

\[\square\]
5 Examples

Let \((P, \mathcal{L})\) be a 4-dimensional projective plane admitting an action of the group \(\Delta = \mathbb{C}^* \cdot \text{SU}_2 \mathbb{C}\). Then it is known that \(\Delta\) fixes an antiflag \((x, W)\) (i.e., a line \(W\) and a point \(x \notin W\)), and that the action of \(\Delta\) on the complex plane \(\mathbb{C}^2\); see [7], Addendum on p. 3; compare also [10], 62.9. If we identify \(P \setminus W\) with \(\mathbb{C}^2\) accordingly, then we have \(x = 0\) and the maximal compact subgroup \(U_2 \mathbb{C} = S_1 \cdot \text{SU}_2 \mathbb{C} \leq \Delta\) leaves every sphere \(S_r = \{q \in \mathbb{C}^2 | |q| = r\}\) invariant. Moreover, it is shown in [7] that the lines passing through the origin 0 in our plane are standard complex lines. By duality, \(\Delta\) acts in the standard way on the set \(\mathcal{L} \setminus (\mathcal{L}_0 \cup \{W\})\) of generic lines, as well.

**PROPOSITION 5.1** Let \((P, \mathcal{L})\) be a 4-dimensional projective plane admitting an action of the group \(\Delta = \mathbb{C}^* \cdot \text{SU}_2 \mathbb{C}\). Then the spheres \(S_r\) described above are hyperunitals isomorphic to the standard hyperunital \(S_1\) in the complex plane, and each of them bounds a compact disk geometry.

**Proof.** First we show that every secant meets \(U\) in a 1-sphere which is a block of the standard unital. By the preceding remarks, this is trivial for the secants passing through 0.

The one-dimensional torus group \(T = \{(z, w) \mapsto (e^{it}z, w) | t \in \mathbb{R}\} \leq U_2 \mathbb{C}\) fixes the complex line \(A = \{0\} \times \mathbb{C}\), and we know that \(A \in \mathcal{L}\). Thus \(T\) is an axial group, compare [10], 23.7. Being commutative, \(T\) has a fixed center \(c\), and \(T \leq \Delta_{[A,c]}\). We have \(c \in W \setminus A\) by [10], 55.28. The nontrivial \(T\)-orbits in the lines \(L \neq W\) passing through \(c\) are circles, and each such circle is contained in some sphere \(S_r\). Therefore, these orbits are blocks of the standard hyperunital on \(S_r\). Now both the system of all spheres \(S_\rho, \rho > 0,\) and the line system \(\mathcal{L}\) are \(\Delta\)-invariant. Therefore, all standard blocks arise in this way from conjugates of \(T\).

This shows at first only that the geometry induced on \(S_r\) is obtained from the standard hyperunital by some fusion of blocks. More precisely, every block \(U \cap G, G \in \mathcal{L}\), is the union of some (trivial or nontrivial) orbits of torus groups conjugate to \(T\), and every torus orbit is contained in some block \(U \cap G\). But if such fusion actually happens, then we get multiple intersections. Indeed, let two distinct torus orbits \(X, Y \subseteq S_r\) be contained in the same line \(L \in \mathcal{L}\). Then for every \(x \in X\) and \(y \in Y\), the standard block \(Z = xy \cap S_r\) meets \(L\) in two points, hence it must also be contained in \(L\). As \(Z\) meets \(X\), these two sets are not orbits of the same 1-torus, and this implies that \(\Delta_L\) contains two distinct 1-tori and \(\dim \Delta_L \geq 2\), a contradiction; remember that the action of \(\Delta\) on the set of generic lines is standard. We have thus proved that the internal structure of \(S_r\) is equal to that of the standard hyperunital.

Next we have to show that every point \(p \in S_r\) belongs to a unique tangent. We may assume that \(p\) belongs to the line \(A\) considered above, which is a secant. The group \(T\) fixes another line \(L \in \mathcal{L}_p\), which must be a tangent, because \(T\) is transitive on \(U \cap L\) and fixes \(p\). The set of secants passing through \(p\) is a connected and \(T\)-invariant subset of \(\mathcal{L}_p \approx S_2\), hence it is homeomorphic to \(\mathbb{R}^2\) or to \(\mathbb{D}_2\), and thus the tangents through \(p\) either
form a (closed or open) 2-disk $T_p$ or there is precisely one tangent. Assume the former case, and recall that $\Delta$ is transitive on the set $S := \mathcal{L} \setminus (\mathcal{L}_0 \cup \{W\})$ of generic lines. Then $\Delta_p$ is transitive on $T_p$, because a collineation mapping one tangent $T_1$ of $S_r$ to another one, $T_2$, must map $T_1 \cap S_r$ to $T_2 \cap S_r$. However, $\Delta_p$ is a 1-torus and cannot be transitive on a disk.

Clearly, $S_r$ bounds a 4-disk $D$ in the affine plane $P \setminus W$. It remains to prove that the intersection $D \cap L$ with a secant is a 2-disk. By what we have seen, $L$ is a tangent of some unital $S_t$ for $t \leq r$ (remember that $\Delta$ is transitive on $S$), hence $L$ contains a unique point of smallest Euclidean distance from $x$. This implies that $r$ is not this smallest distance. Now from $L \setminus W \approx \mathbb{R}^2$ and $S_r \cap L \approx S_1$ we infer that the disk in $L$ bounded by $S_r \cap L$ must be equal to $D \cap L$.

Examples of planes with the properties needed here have been constructed by P. Sperner [11]. In these planes, the unitals considered here are polar unitals, i.e., each unital consists of the absolute points of some polarity, see [8].

Our preceding result now provides us with a variety of possibilities for the construction of planes. We may start from the complex projective plane and substitute a suitable finite or infinite configuration of pairwise disjoint compact disks with disks taken from one or several different Sperner planes. In this way, it is possible to construct planes whose automorphism group is any given finite subgroup of Spin$_l$ or any given crystallographic subgroup of $\mathbb{R}^4 \cdot SU_2 \mathbb{C}$. We may also start from the complex plane or a Sperner plane and consider a nested sequence of compact disks $D_k$ invariant under the same group $\Delta$. Then the domains $D_k \setminus D_{k+1}$ may be taken from distinct planes for distinct values of $k$.

**Concluding Remarks.** a) In the cases $l \in \{4, 8\}$, Proposition 5.1 has an analogue, which we do not formulate as a proposition, because no planes satisfying the necessary assumptions are known except the classical planes. Here the group $\Delta$ may be replaced with $\mathbb{R} \times$ Spin$_{l+1}$, and we assume that the factor $\mathbb{R}$ consists of homologies. It is known ([10], 62.9) that the action of the maximal compact subgroup $\Phi = $ Spin$_{l+1} \leq \Delta$ on the point set is the classical one, hence $\Delta$ is transitive on generic points and on generic lines.. More information has been obtained by Priwitzer [9]: The lines through the origin are classical lines. The stabilizer $\Phi_L$ of a generic line $L$ is isomorphic to SO$_4 \mathbb{R}$ for $l = 4$ and to Spin$_7$ for $l = 8$, and its orbits on $L$, apart from two fixed points, are homeomorphic to $S_{l-1}$. This fact together with its dual suffices to carry through the above proof.

b) Theorem 3.4 might save a little work in future attempts at the construction of planes admitting the group Spin$_{l+1}$, because it can guarantee the existence of intersection points. In Sperner’s existence proof [11] indeed some, comparatively small, amount can be saved in this way.

c) A hyperunital in a compact projective plane certainly contains some information about the plane. However, our result shows that one must not expect too much. In fact, it demonstrates that both the ‘interior’ geometry (the line system on the compact disk bounded by the unital) and the exterior geometry (the complement of the disk) may be changed without changing the unital.
References


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